exit circumference. The number of vortex pairs corresponds with the number of perturbations.

2) The combination of discrete perturbations along the exit circumference and swirling results in a flow structure that consists of a central swirling jet surrounded by vortices with the opposite sense of rotation. The number of outer vortices again corresponds with the number of perturbations.

3) For swirling jets with four or more perturbations, the effect is confined to a few diameters downstream from the exit.

4) The swirling jet with three perturbations develops a stable, three-lobed structure that persists farther than 10 diameters downstream from the exit.

5) The swirling jet with two perturbations develops a stable, two-lobed structure that persists farther than 10 diameters from the exit.

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Natural Frequencies of Cross-Ply Laminated Panels with Matrix Cracks

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Introduction

MATRIX cracks appear in transverse layers of cross-ply laminates subjected to a relatively low tension. These cracks, called tunneling cracks, are perpendicular to the direction of the tensile load and parallel to the general fiber direction in the corresponding layer. Usually these cracks almost instantaneously propagate throughout the thickness of the layer and form a system of regularly spaced cracks. Such cracks were observed both in polymeric composites[1,2] as well as in ceramic matrix composites.[3,4] A number of theoretical models of the behavior of cross-ply laminates with matrix cracks in transverse layers have been developed. Most of these studies have been listed in a recent paper.[5] Han and Hahn expanded the analysis to the case where tunneling cracks exist in both transverse and longitudinal layers.[6]

Cross-ply laminates with matrix cracks behave in a manner similar to bimodular materials as a result of a different stiffness that depends on the cracks being open or closed. Therefore, the response in tension and in compression differs, although the borderline between the different responses depends on the exact value of the crack closing strain. This complication of the analysis makes it necessary to recall extensive work on mechanics of bimodular materials. Although a comprehensive review is outside the scope of this Note, mentioned here are the papers of Bert et al.,[7] Rebello et al.,[8] and Gordaninejad.[9]

The present Note outlines the analysis of free vibrations of a cross-ply plate with tunneling matrix cracks in all layers. The closed-form analytical solution is obtained in the paper utilizing the assumption that the cracks are closed under compression and the expressions for the engineering constants of cross-ply laminates with matrix cracks suggested by Han and Hahn.[6] The changes in natural frequencies considered in this Note are caused by a reduction of the stiffness. The effect of damping, which increases with cracking, is not accounted for in the present solution.

Analysis

Consider free vibrations of a thin cross-ply laminated plate with tunneling matrix cracks in the longitudinal and transverse layers (hereafter, longitudinal layers are oriented in the x direction and transverse layers are in the y direction). The plate can be asymmetrically laminated and the distribution of matrix cracks (crack density) can vary from layer to layer, although it is assumed here that this distribution is independent of the in-plane coordinates.

The analysis is based on the following assumptions:

1) The stiffness of a layer in the fiber direction is not affected by tunneling cracks. This reflects the observation that the planes of the cracks are parallel to the fibers of the corresponding layer and the fiber stiffness is the major contributor to the longitudinal stiffness.

2) Cracks are assumed to close under compression. Note that a crack closing strain may be estimated using the ratio of the crack opening displacement to the crack spacing (the latter is defined as a distance between the planes of parallel cracks). However, tunneling cracks reach saturation at the spacing, which is larger than the layer thickness, as was shown for both polymeric and ceramic matrix composites.[10,11] Therefore, the error introduced by the assumption that cracks close under compression cannot be large.

Although the longitudinal modulus of elasticity and the Poisson ratios of a compressed layer with closed cracks are recovered, the in-plane shear modulus depends on friction between the faces of closed cracks that has not been reported, to the best knowledge of the authors. If the friction coefficient is assumed negligible, the shear modulus is unaffected by the cracks being open or closed.

Small-amplitude free vibrations of the panel are analyzed by assumption that they do not result in additional damage. Accordingly, we consider the motion with preexisting matrix cracks, in which case thermal residual stresses do not affect the stiffness.[6] The expressions for engineering constants utilized in the subsequent calculations are based on the solution of Han and Hahn.[6] Based on these engineering constants, one can formulate the matrices of transformed reduced stiffnesses, as discussed next. For convenience, let us denote $z_1 = \max(z_x, z_y)$ and $z_2 = \min(z_x, z_y)$, where z_x and z_y are the locations of the neutral planes corresponding to zero axial strains ε_x and ε_y, respectively. It is assumed that during the part of the motion cycle considered next the cracks are open in all layers of the section of the laminate located within $z_1 < z < h/2$, whereas in the part corresponding to $-h/2 < z < z_2$ the cracks in both longitudinal and transverse layers are closed.

Matrix of Reduced Stiffnesses Within $z_1 < z < h/2$

The cracks are open in both longitudinal and transverse layers. Therefore, reduced stiffnesses Q'_{11}, Q'_{12}, Q'_{22}, and Q'_{66} can be determined in terms of the engineering constants E'_x, ν'_{xy}, E'_y, and G'_{xy} calculated according to the theory of Han and Hahn.[6] The engineering constants referred to in this Note are the average values for the material, rather than the constants for an individual layer. Notably, these constants depend on the density of cracks in several adjacent layers. Accordingly, they may vary throughout the depth of the panel, as is the case if matrix cracks were caused by bending.

Matrix of Reduced Stiffnesses Within $z_2 < z < z_1$

In this interval the cracks are closed in either longitudinal or transverse layers, whereas the cracks in the perpendicular layers are

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open. Consider, for example, the situation where cracks are closed in the longitudinal layers and open in the transverse layers. In this case the modulus of elasticity in the y direction is recovered, and it corresponds to the value for the intact material, i.e., E_y. Accordingly, the new values of reduced stiffnesses Q''_{11}, Q''_{12}, Q''_{22}, and Q''_{66} can be found using adjusted values of engineering constants E''_x, ν''_{xy}, G''_{xy}, and E_y where the quantities with double prime represent the constants in the case where cracks are confined to transverse layers.[6]

Matrix of Reduced Stiffnesses Within $-h/2 < z < z_2$

In this part of the cross section, the cracks are closed, and all engineering constants, except for the shear modulus, correspond to those for the intact material. Reduced stiffnesses Q_{11}, Q_{12}, Q_{22}, and Q_{66} are immediately available.

Analysis of Free Vibrations

During vibrations, a panel that could be symmetric prior to damage becomes asymmetric because of the presence of matrix cracks. In a cross-ply panel $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$. Accordingly, equations for free transverse vibrations of a thin panel are

$$
\begin{aligned}
&A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} \\
&\qquad - (B_{12} + 2B_{66})w_{,xyy} = 0 \\
&(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{22}v_{,yy} - (B_{12} + 2B_{66})w_{,xxy} \\
&\qquad - B_{22}w_{,yyy} = 0 \\
&D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - B_{11}u_{,xxx} \\
&\qquad - (B_{12} + 2B_{66})u_{,xyy} - (B_{12} + 2B_{66})v_{,xxy} - B_{22}v_{,yyy} = -\rho w_{,tt}
\end{aligned}
\tag{1}
$$

In-plane inertial terms are neglected in these equations, as is customary in similar problems where the frequencies of in-plane vibrations are much higher than those of transverse motion. Displacements u, v, and w are in the directions of x, y, and z axes, respectively. The mass per unit surface area is denoted by ρ, and t is time. Extensional, coupling, and bending stiffness coefficients are determined from

$$
[A_{ij}, B_{ij}, D_{ij}] = \int_{-h/2}^{z_2} Q_{ij}[1, z, z^2]\,\mathrm{d}z + \int_{z_2}^{z_1} Q''_{ij}[1, z, z^2]\,\mathrm{d}z + \int_{z_1}^{h/2} Q'_{ij}[1, z, z^2]\,\mathrm{d}z \tag{2}
$$

where the elements of the matrix of reduced stiffnesses Q_{ij} are defined for each integral in the right side according to the preceding discussion.

Equations of motion should satisfy the boundary conditions. In the present problem they correspond to simple support:

$$
\begin{aligned}
x = 0, x = a&: \quad w = M_x = N_x = v = 0 \\
y = 0, y = b&: \quad w = M_y = N_y = u = 0
\end{aligned}
\tag{3}
$$

where the lengths of the panel in the x and y directions are denoted by a and b, respectively.

It is immediately recognized that conditions (3) are present corresponding to the case where the edges are supported by beams that are very stiff in the plane of the web but have negligible torsional and out-of-web plane stiffness. Such situations can be found in applications where supports are in the form of blade or other open-profile stiffeners.

The expressions for the stress resultants and stress couples in boundary conditions (3) represented in terms of displacements are

$$
\begin{aligned}
N_x &= A_{11}u_{,x} + A_{12}v_{,y} - B_{11}w_{,xx} - B_{12}w_{,yy} \\
M_x &= B_{11}u_{,x} + B_{12}v_{,y} - D_{11}w_{,xx} - D_{12}w_{,yy}
\end{aligned}
\tag{4}
$$

Expressions for N_y and M_y are obtained by analogy.

The boundary conditions are identically satisfied if displacements are represented by the following expressions:

$$
\begin{aligned}
u &= U\cos(m\pi x/a)\sin(n\pi y/b)\exp(i\omega t) \\
v &= V\sin(m\pi x/a)\cos(n\pi y/b)\exp(i\omega t) \\
w &= W\sin(m\pi x/a)\sin(n\pi y/b)\exp(i\omega t)
\end{aligned}
\tag{5}
$$

where U, V, and W are amplitudes of the corresponding displacements, the integers m and n define the mode shape of motion, $i = (-1)^{1/2}$, and ω is the natural frequency.

The substitution of Eqs. (5) into equations of motion (1) yields the set of homogeneous algebraic equations

$$
\begin{gathered}
k_{11}U + k_{12}V + k_{13}W = 0, \qquad k_{12}U + k_{22}V + k_{23}W = 0 \\
k_{13}U + k_{23}V + \left(k_{33} - \rho\omega^2\right)W = 0
\end{gathered}
\tag{6}
$$

The coefficients k_{ij} are omitted here for brevity.

The natural frequency $\omega = \omega(m, n)$, and the ratios U/W and V/W could be available from the system of equations (6) if the locations of the neutral planes for longitudinal and transverse strains were defined. If the amplitude ratios are known, these locations are available from

$$
\varepsilon_x = u_{,x} - z_x w_{,xx} = 0, \qquad \varepsilon_y = v_{,y} - z_y w_{,yy} = 0 \tag{7}
$$

Accordingly, $z_x = [a/(m\pi)](U/W)$ and $z_y = [b/(n\pi)](V/W)$. In the present problem the location of the neutral planes is not affected by the in-plane coordinates, i.e., by x and y. A similar conclusion was obtained in the previous work of Bert and his collaborators on bimodular materials.[7,8] However, this conclusion is only applicable in the case of linear vibrations where the interaction between different modes is absent. It is also emphasized that the locations of the neutral planes, i.e., z_x and z_y, differ for different mode shapes of vibrations.

The closed-form solution is available using the following approach. The coefficients k_{ij} in Eqs. (6) are dependent on the elements of the matrices A, B, and D that can be expressed in terms of the reduced stiffnesses and coordinates z_x and z_y. Accordingly, it is possible to derive the ratios $U/W = f_{mn}(z_x, z_y)$ and $V/W = g_{mn}(z_x, z_y)$ from the first two Eqs. (6). Subsequently, the values of z_x and z_y can be determined from the system of equations that follows from Eqs. (7):

$$
(m\pi/a)z_x = f_{mn}(z_x, z_y), \qquad (n\pi/b)z_y = g_{mn}(z_x, z_y) \tag{8}
$$

If the crack distribution about the middle plane of a symmetrically laminated panel is symmetric, the frequency can immediately be found from the third Eq. (7). However, if the crack distribution is asymmetric, the frequency depends on the reference surface $z = h/2$ being subject to tension or compression. Therefore, we have to find $\omega_t(m, n)$ and $\omega_c(m, n)$ for the former and latter cases. Subsequently, the actual vibration frequency is obtained from

$$
\omega^{-1}(m, n) = \left[\omega_t^{-1}(m, n) + \omega_c^{-1}(m, n)\right]/2 \tag{9}
$$

Particular Case: Large-Aspect-Ratio Plate with Matrix Cracks in Transverse Layers

Consider a large-aspect-ratio cross-ply laminated plate where $b \gg a$. In this case the mode shape of vibration is cylindrical, i.e., all derivatives with respect to the y coordinate can be omitted in Eqs. (1) and (4). The nondimensional frequency ω_{nd} that represents a ratio of the frequency of a damaged plate to the frequency of the intact counterpart is obtained as

$$
\omega_{\mathrm{nd}} = \left[\left(D_{11} - B_{11}^2/A_{11}\right)/\left(D_{11i} - B_{11i}^2/A_{11i}\right)\right]^{\frac{1}{2}} \tag{10}
$$

where the subscript i identifies the stiffnesses of the intact plate.

It can be shown that the location of the neutral plane $z_x = B_{11}/A_{11}$. Substituting the expressions for A_{11} and B_{11} in terms of the reduced stiffnesses Q''_{11} and Q_{11} and z_x, one obtains a quadratic equation:

$$
az_x^2 + bz_x + c = 0 \tag{11}
$$

Table 1 Nondimensional coordinate of the middle plane as a function of the nondimensional matrix crack spacing

s_n	z_{nx}
4	−0.0178
8	−0.00924
12	−0.00624
16	−0.00471
20	−0.00355

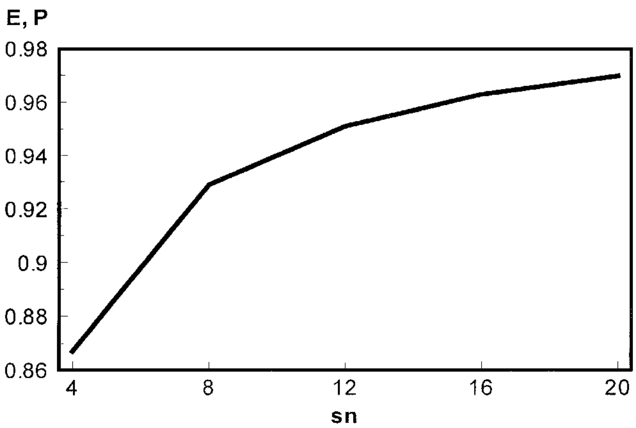

Fig. 1 Nondimensional modulus of elasticity *E* and the normalized Poisson ratio *P* as functions of the nondimensional matrix crack spacing s_n. Two curves practically coincide.

where

$$a = (Q_{11} - Q''_{11})/2, \qquad b = (Q_{11} + Q''_{11})h/2$$

$$c = \tfrac{1}{2}(Q_{11} - Q''_{11})(h/2)^2 \tag{12}$$

For a given distribution of matrix cracks, the solution of Eq. (11) yields the location of the neutral plane. Subsequently, the stiffness coefficients can be calculated and the nondimensional frequency found from Eq. (10).

Numerical Example

The plate considered in this example has a large aspect ratio. Prior to damage, the plate was symmetrically laminated. The material considered in these examples is SiC/CAS,[12] which has the moduli of elasticity in the fiber and transverse directions equal to 121 and 112 GPa, respectively, and the major Poisson ratio equal to 0.20. The cracks are limited to transverse layers, and their spacing is uniform throughout the thickness of the plate. According to the measurements of Erdman and Weitsman,[11] the matrix crack saturation in transverse layers of balanced SiC/CAS cross-ply composites occurs when the spacing is several times the thickness of the layer.

The nondimensional modulus of elasticity $E = E''_x / E_{xi}$ and the normalized Poisson ratio $P = \nu''_{xy} / \nu_{xyi}$ are shown as functions of the nondimensional matrix crack spacing s_n in Fig. 1 (s_n is defined as a ratio of the crack spacing to the thickness of the layer). As follows from this figure, a decrease of the elasticity modulus becomes significant (close to 14%) only if the cracks approach saturation. The nondimensional Poisson ratio also decreases, as could be expected because while the stiffness in the x direction becomes smaller with the accumulation of cracks the stiffness in the y direction remains unaffected by cracking. The curves of E and P practically coincide. This serves as an indirect justification of the assumption already introduced that the stiffness in the direction parallel to the crack planes is not affected by the presence of damage.

During vibrations, the location of the neutral plane for the axial strains in the x direction is very close to the middle plane of the laminate, even when the cracks reach saturation. This is reflected in Table 1, which was generated for the case where the surface $z = h/2$ is subject to tension (accordingly, $z_x < 0$). The value z_{nx} in this table is defined as z_x / h, i.e., even as the cracks reach saturation, the neutral plane moves by less than 2% from the middle plane of the plate. It was observed that the frequencies obtained by assumption that the neutral and middle planes coincide were practically identical to those calculated without this simplification.

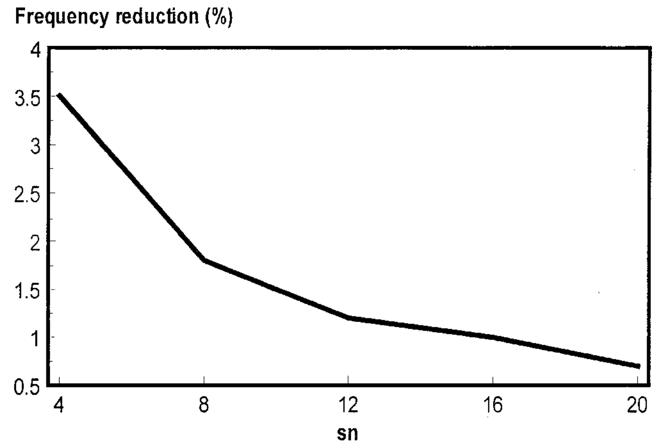

Fig. 2 Percentage of the frequency reduction as a function of the nondimensional matrix crack spacing.

The deviation of the fundamental frequency of vibrations from the frequency for the intact plate is shown in Fig. 2 as a function of the nondimensional matrix crack spacing. As follows from Fig. 2, the changes of the natural frequency remain below 4%. The observation that the changes of the stiffness caused by matrix cracking are more pronounced than the changes of the frequencies could be anticipated because the frequency is proportional to a square root of the stiffness.

Even if tunneling matrix cracks exist in longitudinal layers, their effect on the natural frequency of a large-aspect-ratio panel considered here is negligible. The reason is that the planes of these cracks are parallel to the x axis. Accordingly, their presence does not affect the stiffness in the x direction.

Conclusions

In conclusion, the Note presents a closed-form solution for the natural frequencies of a cross-ply laminated panel with tunneling matrix cracks. As follows from the numerical analysis, the neutral planes of the panel are located very close to the middle plane. Therefore, it is possible to estimate the frequencies by assumption that the cracks are open in the half of the cross section located on the side of the middle plane subjected to tension and closed on the opposite side. The effect of tunneling matrix cracks on the stiffness is much larger than their effect on the natural frequency.

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Probabilistic Finite Element Model Updating Using Random Variable Theory

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Introduction

THE finite element (FE) method is a popular technique used to determine structural responses of complex structures. However, modal surveys performed on actual hardware do not always agree with analytically predicted natural frequencies and mode shapes. The goal of model updating/refinement, therefore, is to correct the deficiencies found within the analytical FE model. There are several classes of model updating schemes. The simplest ones are direct updating in which mass and/or stiffness matrices are updated in a single iteration, such as in Ref. 1. Another class of schemes utilize sensitivity methods.[2] A detailed overview/survey of model updating techniques in structural dynamics is provided by Imregun and Visser[3] and Mottershead and Friswell.[4]

In the past, dynamic systems have been assumed to have well-defined structural properties or to have small variations in them. A deterministic analysis is then usually performed to obtain the desired result, that is, static or dynamic analysis. However, systems may not always be well defined, and a stochastic analysis must be performed. Modal parameter uncertainty, for example, may arise from randomness in the structural properties (due to manufacturing variability), from measurement errors (e.g., noisy sensors), statistical variations in the measurement process (e.g., inaccurate data acquisition system), and/or variations with all of the myriad of system identification algorithms. Repeated experimentation will also exhibit modal parameter variations.

These modal parameter variations will be explored to improve the robustness characteristics of current model updating methodologies. As a result, all uncertainty will be quantified by treating the stiffness (or, in this case, flexural rigidity *EI*) as random with an associated probability density function. Because of the mathematics involved, a caveat that should be stated is that the number of experimental modes measured is required to be the same as the number of unknowns in the problem, thus allowing for a unique solution. Note that this assumption is very limiting when relating to any realistic problem, but the intent of the Note is to show the method. Finally, note that Papadopoulos and Garcia[5] applied this technique using noise-free simulated data on a cantilever Euler–Bernoulli beam and found that the method is able to correct the errors in the initial FE model. This Note adds to the previous work by evaluating the technique using actual experimental data from a similar cantilever beam. In addition, the underlying methodology for this technique is a modification of the damage detection theory formulated by Papadopoulos and Garcia.[6,7]

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Theoretical Development

The main development of this technique may be found in Ref. 5 and in related reference materials.[6,7] On considering the free vibration eigenvalue problem for an Nth-order undamped dynamic system and assuming that the initial analytical uncorrected model is related to the experimental model by an amount Δ, we arrive at the principle equation

$$\Delta\lambda_i\{\phi^a\}_i^T[M^a]\{\phi^{\mathrm{ex}}\}_i = \sum_{j=1}^{L}\alpha_j\{\phi^a\}_i^T\left[K_j^e\right]^a\{\phi^{\mathrm{ex}}\}_i \tag{1}$$

for $i = 1, 2, 3, \ldots, N$, where quantities with a superscript a denote initial analytical data and ex denotes experimental data. $[M^a]$ represents the $(N \times N)$ symmetric analytical mass matrix, $[K_j^e]$ denotes the sparse analytical jth element stiffness submatrix, λ is the ith mode scalar eigenvalue, $\{\phi\}_i$ is the ith mode eigenvector or mode shape of size $(N \times 1)$, and α_j represents the jth element stiffness reduction factor (SRF). The SRFs indicate the amount of stiffness correction required for each jth element to accommodate the non-agreement between the analytical and experimental modal parameters. For example, a value of -0.5 indicates that the elemental stiffness needs to be reduced by 50%, and a value of $+0.5$ indicates that it needs to be increased by 50%. L denotes the total number of structural elements comprising the system, and T denotes a matrix transpose.

Equation (1) represents a set of N simultaneous linear equations with L unknowns that can be written in the form of

$$[A]_{(N\times L)}\{q\}_{(L\times 1)} = \{b\}_{(N\times 1)} \tag{2}$$

whose elements are given as

$$A_{ij} = \{\phi^a\}_i^T\left[K_j^e\right]^a\{\phi^{\mathrm{ex}}\}_i$$

$$q_j = \alpha_j$$

$$\begin{aligned} b_i &= \Delta\lambda_i\{\phi^a\}_i^T[M^a]\{\phi^{\mathrm{ex}}\}_i \\ &= \left(\lambda_i^{\mathrm{ex}} - \lambda_i^a\right)\{\phi^a\}_i^T[M^a]\{\phi^{\mathrm{ex}}\}_i \end{aligned} \tag{3}$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, L$. The vector $\{q\}$ in Eq. (2) represents the SRFs for each structural element. In general, the number of experimentally measured modes N will be less than the total number of individual structural elements L of the system. Therefore, the matrix $[A]$ will be rectangular and noninvertible, that is, a set of underdetermined systems of equations. For the special case when $N = L$, there will be a unique solution because $[A]$ will be square and invertible. This Note will focus on the ideal case when $N = L$ (although this will not be the case for any realistic problems), thereby permitting a unique solution. Note that this assumption is very limiting when relating to any realistic problem, but the intent of the Note is to show the method. The quantities in Eq. (2) will be treated as random variables assumed to have normal distributions, where the expected value and covariance matrix of the unknowns $\{q\}$ are found in Ref. 5.

Results

To illustrate the proposed method, the theory will be applied to the model correction of an aluminum cantilever Euler–Bernoulli beam using experimental data. The beam used in the experimental